IR issues in cosmological perturbation - constraints on quantum state -

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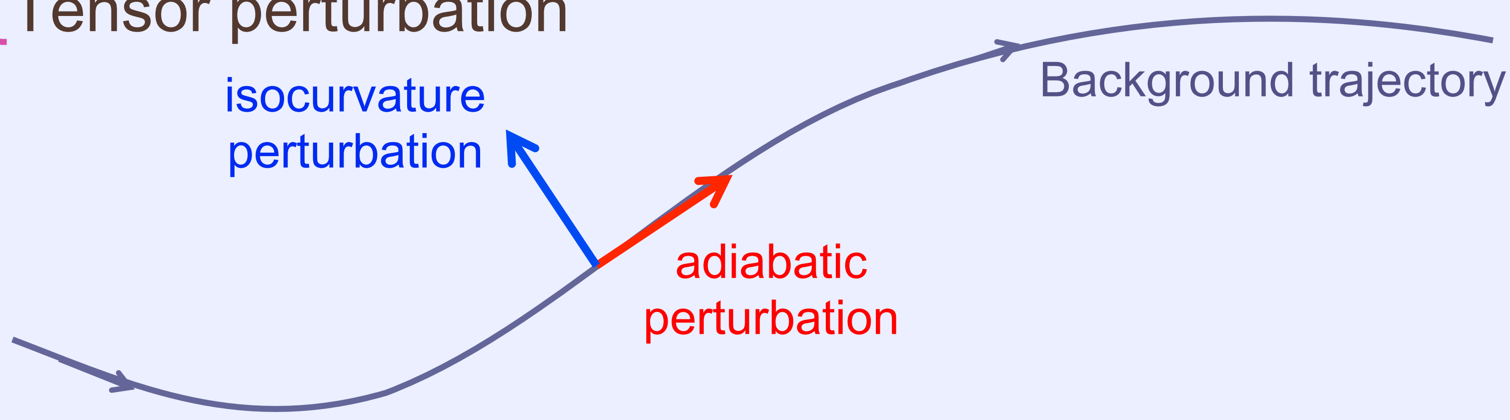
(JCAP to be published: arXiv:1103.1251)
PTP to be published: arXiv:1009.2947,
Phys. Rev. D82:121301, 2010: arXiv:1007.0468
PTP 122: 779, 2009: arXiv:0902.3209

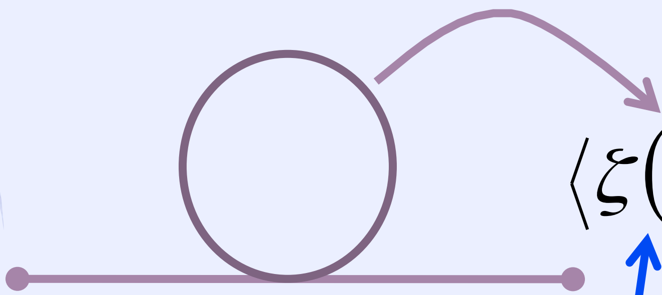
Various IR issues

- IR divergence coming from k-integral
- Secular growth in time $\propto (HT)^n$

- Adiabatic perturbation,
  - which can be locally absorbed by the choice of time slicing.
- Isocurvature perturbation
  - ≈ field theory on a fixed curved background
- Tensor perturbation



§IR divergence in single field inflation



Factor coming from this loop:

$$\langle \zeta(y)\zeta(y) \rangle \approx \int d^3k \; \underline{P(k)} \approx \log\left(aH / k_{\min}\right)$$

$$\boxed{\propto 1/k^3}$$

curvature perturbation in co-moving gauge.

scale invariant spectrum
- no typical mass scale

$$\begin{cases} h_{ij} = e^{2N+2\zeta}\left(\delta_{ij} + h_{ij}\right) \\ \delta\phi = 0 \end{cases}$$

Transverse traceless

Special property of single field inflation

Yuko Urakawa and T.T., PTP122: 779 arXiv:0902.3209

- In conventional cosmological perturbation theory, gauge is not completely fixed.

Time slicing can be uniquely specified: $\delta\phi = 0$ OK!

but spatial coordinates are not.

$$h_j^j = 0 = h_{i\ ,j}^{j}$$

Residual gauge d.o.f.

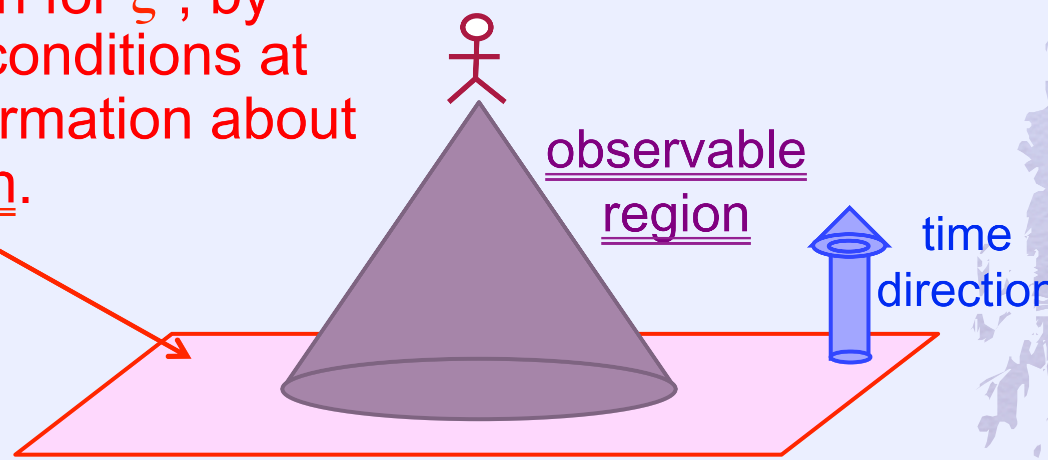
$$\delta_g h_{ij} = \xi_{i,j} + \xi_{j,i}$$

Elliptic-type differential equation for ξ^i.

$$\Delta \xi^i = \cdots$$

Not unique locally!

- To solve the equation for ξ^i, by imposing boundary conditions at infinity, we need information about <u>un-observable region</u>.



Basic idea of the proof of IR finiteness in single field inflation

- The local spatial average of ζ can be set to 0 identically by an appropriate gauge choice.
- Even if we choose such a local gauge, the evolution equation for ζ formally does not change, and it is hyperbolic. So only the interaction vertices inside the past light cone are relevant.
- Therefore, IR effect is completely suppressed as long as we compute ζ in this local gauge.

However, we later noticed that the above argument is true only when correlation functions of ζ are free from divergence at the initial time, which is not in general guaranteed.

Genuine gauge-invariant quantities

- If we evaluate genuine gauge-invariant quantities, we should obtain finite results whatever gauge we may use.

A genuine gauge-invariant quantity:

Correlation functions for 3-d scalar curvature on ϕ =constant slice.

$$\langle R(\boldsymbol{x}_1)\, R(\boldsymbol{x}_2)\rangle$$

But coordinate values do not have gauge invariant meaning.

(Giddings & Sloth 1005.1056)
(Byrnes et al. 1005.33307)

$$\boldsymbol{x}(\boldsymbol{X}_A, \lambda{=}1) = \boldsymbol{X}_A + \delta\boldsymbol{x}_A$$

Specify the position by solving geodesic Eq. $D^2x^i/d\lambda^2 = 0$

with initial condition $Dx^i/d\lambda\big|_{\lambda=0} = X^i$.

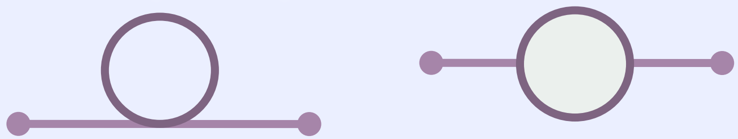
Then, use X^i to specify the position.

$${}^gR(\boldsymbol{X}_A) := R(\boldsymbol{x}(\boldsymbol{X}_A, \lambda{=}1)) = R(\boldsymbol{X}_A) + \delta\boldsymbol{x}_A \nabla R(\boldsymbol{X}_A) + \ldots$$

$\langle {}^gR(\boldsymbol{X}_1)\, {}^gR(\boldsymbol{X}_2)\rangle$ should be genuine gauge invariant.

Translation invariance of the vacuum state takes care of the ambiguity in the choice of the origin.

One-loop 2-point function at the leading slow-roll exp.



- No interaction term in the evolution equation at $O(\varepsilon^0)$ in flat gauge.

◎flat gauge → $\delta\phi = 0$ gauge ◎$R(X_A) \sim e^{-2\zeta}\Delta\zeta$ ◎$R \rightarrow {}^gR$

$$\langle {}^gR(X_1)\,{}^gR(X_2)\rangle \propto \langle \zeta_I{}^2\rangle \int d(\log k) k^3 \Big[\Delta(\boldsymbol{D}^2 u_{\boldsymbol{k}}(X_1))\Delta(u^*_{\boldsymbol{k}}(X_2)) + 2\Delta(\boldsymbol{D}u_{\boldsymbol{k}}(X_1))\Delta(\boldsymbol{D}u^*_{\boldsymbol{k}}(X_2)) + \Delta(u_{\boldsymbol{k}}(X_1))\Delta(\boldsymbol{D}^2 u^*_{\boldsymbol{k}}(X_2))\Big] + \text{c.c.}$$

$+$ (manifestly finite pieces)

where $\zeta_I = u_{\boldsymbol{k}}\, a_{\boldsymbol{k}} + u^*_{\boldsymbol{k}}\, a^\dagger_{\boldsymbol{k}}$ $\quad \boldsymbol{D} := \partial_{\log a} - (\boldsymbol{x}\cdot\boldsymbol{\nabla})$

- IR divergence from $\langle \zeta_I{}^2\rangle$ exists in general.

However, the integral vanishes for the Bunch-Davies vacuum state.

$$u_{\boldsymbol{k}} = k^{-3/2}(1 - ik/aH)e^{ik/aH} \Longrightarrow \boldsymbol{D}u_{\boldsymbol{k}} = k^{-3/2}\,\partial_{\log k}(k^{3/2}u_{\boldsymbol{k}})$$

Then
$$\langle {}^gR(X_1)\,{}^gR(X_2)\rangle^{(4)} \propto \langle \zeta_I{}^2\rangle \times \int d(\log k)\; \partial^2_{\log k}\left[\Delta(k^{3/2}u_{\boldsymbol{k}}(X_1))\Delta(k^{3/2}u^*_{\boldsymbol{k}}(X_2))\right] + \text{c.c.}$$

- To remove IR divergence, the positive frequency function corresponding to the vacuum state is required to satisfy $\boldsymbol{D}\,u_{\boldsymbol{k}} = k^{-3/2}\,\partial_{\log k}(k^{3/2}u_{\boldsymbol{k}})$.

IR regularity requests scale invariance!

Summary of what we found

1) To avoid IR divergence, the initial quantum state must be “scale invariant/Bunch Davies” in the slow roll limit.

“Wave function must be homogeneous in the residual gauge direction”

2) *To the second order of slow roll*, <u>a generalized condition of “scale invariance”</u> to avoid IR divergence was obtained, and found to be consistent with the EOM and normalization.

3) Computation that assumes adiabatic vacuum (e.g. Giddings and Sloth) finds no IR divergence. This means that our generalized condition of scale invariance should be compatible with the adiabatic vacuum choice.

Consistency relation

In the squeezed limit, 3pt fn is given by

$$\left\langle \zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\right\rangle \approx -\delta\left(\boldsymbol{k}_1+\boldsymbol{k}_2+\boldsymbol{k}_3\right)\left(n_s-1\right)P_{\boldsymbol{k}_1}P_{\boldsymbol{k}_2}$$ for $k_1 << k_2, k_3$

Super-horizon long wavelength mode k_1 should be irrelevant for the short wavelength modes k_2, k_3.

The only possible effect of k_1 mode is that it modifies the proper wave numbers corresponding to k_2, k_3.

Let's consider ζ in geodesic normal coordinates $\boldsymbol{X}\sim e^{\zeta_{\boldsymbol{k}_1}}\boldsymbol{x}$

$${}^g\zeta_{\boldsymbol{k}_a} \approx \zeta_{\boldsymbol{k}_a} + \zeta_{\boldsymbol{k}_1}\left(\boldsymbol{k}_a\cdot\partial_{\boldsymbol{k}_a}+\frac{3}{2}\right)\zeta_{\boldsymbol{k}_a}+\cdots$$, neglecting tensor modes

$\langle\zeta_{k_1}\,{}^g\zeta_{k_2}\,{}^g\zeta_{k_3}\rangle \approx 0 \Longrightarrow \left\langle \zeta_{\boldsymbol{k}_1}\zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\right\rangle \approx -P_{\boldsymbol{k}_1}\left(\partial_{\log k_2}+\partial_{\log k_3}-3\right)\left\langle \zeta_{\boldsymbol{k}_2}\zeta_{\boldsymbol{k}_3}\right\rangle$

$\because$ ${}^g\zeta_{k_2}$ and ${}^g\zeta_{k_3}$ are not correlated with ζ_{k_1}.

This derivation already indicates that the leading term in the squeezed limit given by the consistency relation vanishes once we consider "*genuine gauge invariant quantities*".

Vanishing 3pt function

$\langle \zeta_{\boldsymbol{k}_1}\, {}^g\zeta_{\boldsymbol{k}_2}\, {}^g\zeta_{\boldsymbol{k}_3} \rangle \approx 0$ In the squeezed limit, 3pt fn vanishes,

(1) for $k_1 << (aL)^{-1} << k_2, k_3$ or (2) for $(aL)^{-1} << k_1 << k_2, k_3$?

size of our observable universe

In the case (1), Fourier mode with such small k_1 cannot be resolved!

But approximate expression for geodesic normal coordinates $\boldsymbol{X} \sim e^{\zeta_{\boldsymbol{k}_1}} \boldsymbol{x}$ is valid only for the case (1).

For extension to the case (2), we have to solve the geodesic equation:

$$\frac{d^2 x^i}{d\lambda^2} = -\Gamma^i{}_{jk} \frac{dx^j}{d\lambda} \frac{dx^k}{d\lambda} \approx -i\left[2(\boldsymbol{k}_1 \cdot \boldsymbol{X}) X^i - X^2 k_1^i\right] \zeta(\boldsymbol{k}_1) e^{i\boldsymbol{k}\cdot\boldsymbol{X}}$$

$\Rightarrow$ Although it's too technical to explain it here, even if we include the above corrections to the relation between, $\boldsymbol{x}$ and $\boldsymbol{X}$, $\langle \zeta_{\boldsymbol{k}_1}\, {}^g\zeta_{\boldsymbol{k}_2}\, {}^g\zeta_{\boldsymbol{k}_3} \rangle \approx 0$ holds in the squeezed limit.

Summary of the last part

- In the squeezed limit bispectrum is known to be given by the power spectrum and spectral index.
- But this applies for the usual ζ, which is not a genuine gauge invariant variable.
- Leading term given by the consistency relation disappears if we use a genuine gauge invariant quantity.
- We suspect that a similar thing may happen for the temperature perturbation in CMB, which should be a genuine gauge invariant quantity.
- Remark: this argument applies only for local-type non-Gaussianity originating from initial adiabatic perturbation, though.

THANK YOU!